

ON J. C. C. NITSCHKE TYPE INEQUALITY FOR ANNULI ON RIEMANN SURFACES

DAVID KALAJ

ABSTRACT. Assume that (\mathcal{N}, \bar{h}) and (\mathcal{M}, φ) are two Riemann surfaces with conformal metrics \bar{h} and φ . We prove that if there is a harmonic homeomorphism between an annulus $\mathcal{A} \subset \mathcal{N}$ with a conformal modulus $\text{Mod}(\mathcal{A})$ and a geodesic annulus $A_\varphi(p, \rho_1, \rho_2) \subset \mathcal{M}$, then we have $\rho_2/\rho_1 \geq \Psi_\varphi \text{Mod}(\mathcal{A})^2 + 1$, where Ψ_φ is a certain positive constant depending on the upper bound of Gaussian curvature of the metric φ . An application for the minimal surfaces is given.

1. INTRODUCTION

1.1. Background and statement of the main result. In order to include some background related to the problem treated in this paper assume that $\mathbf{A}(a, b)$ is the annulus $\{x \in \mathbf{R}^n : a < |x| < b\}$ in the Euclidean space \mathbf{R}^n , $n \geq 2$. Fifty years ago J. C. C. Nitsche [19], studying the minimal surfaces and inspired by radial harmonic homeomorphism between annuli, by using Harnack inequality for positive harmonic functions in \mathbf{R}^n proved that the existence of an Euclidean harmonic homeomorphism between annuli $\mathbf{A}(r, 1)$ and $\mathbf{A}(\rho, 1)$ is equivalent with an inequality $\rho < \rho_r$, where ρ_r is a positive constant smaller than 1. Then he conjectured that for $n = 2$

$$(1.1) \quad \rho_r = \frac{2r}{1 + r^2}.$$

This conjecture is solved recently in positive by Iwaniec, Kovalev and Onninen in [4]. Some partial results have been obtained previously by Lyzzaik [18], Weitsman [24] and the author [15]. On the other hand in [5] and in [14] is treated the same problem for the harmonic mappings w.r.t the hyperbolic and Riemann metric in two-dimensional hyperbolic space and in two-dimensional Riemann sphere respectively. The n -dimensional generalization of J. C. C. Nitsche conjecture is

$$(1.2) \quad \rho_r = \frac{nr}{n - 1 + r^n}$$

and is inspired by the radial harmonic mapping

$$(1.3) \quad f(x) = \left(\frac{1 - r^{n-1}\rho}{1 - r^n} + \frac{r^{n-1}\rho - r^n}{(1 - r^n)|x|^n} \right) x$$

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between annuli $\mathbf{A}(r, 1)$ and $\mathbf{A}(\rho, 1)$ (c.f. [13]). In [14, 13] the author treated the three-dimensional case and obtained an inequality for Euclidean harmonic mappings between annuli on \mathbf{R}^3 . The last conjectured inequality for $n \geq 3$ remains an open problem.

The Nitsche phenomena is rooted in the theory of doubly connected minimal surfaces (cf. [4]) and in the existence problem of maps between annuli of the complex plane, Riemann surface or Euclidean space with the least Dirichlet energy (cf. [8], [16], [21]).

In this paper we consider harmonic mappings between certain annuli of the Riemannian surfaces and study J. C. C. Nitsche type problem for the harmonic mappings between geodesic annuli of Riemannian surfaces. By (\mathcal{M}, \wp) we denote a Riemann surface with the conformal metric $\wp(w)|dw|$, whose Gaussian curvature $K = K_\wp(w)$ is bounded from the above by a constant $\pm\kappa^2$ ($\sup K_\wp(w) = \kappa^2$ or $\sup K_\wp(w) = -\kappa^2$). By $B_\wp(p, \rho_p)$ we denote the geodesic ball with the center at $p \in \mathcal{M}$ which is disjoint from the cut-locus of p with the radius $\rho_p > 0$ (which for $\sup K = \kappa^2$ satisfies the additional condition $\rho_p \leq \pi/(2\kappa)$). The geodesic annulus is defined by $A_\wp(p, \rho_1, \rho_2) = B_\wp(p, \rho_2) \setminus \overline{B_\wp(p, \rho_1)}$. (We refer to section 2.1 for more details concerning the above concepts.) The following is the main result of this paper

Theorem 1.1 (The main result). *If there exists a harmonic homeomorphism between an annulus $A(r_1, r_2) \subset \mathbf{R}^2$ and a geodesic annulus $A_\wp(p, \rho_1, \rho_2)$ of the Riemann surface (\mathcal{M}, \wp) , then we have*

$$(1.4) \quad \frac{\rho_2}{\rho_1} \geq \Psi_\wp \log^2 \frac{r_2}{r_1} + 1,$$

where

$$(1.5) \quad \Psi_\wp = \begin{cases} \frac{\sinh[\kappa\rho_1]}{2\kappa\rho_1}, & \text{if } \sup_{w \in \mathcal{M}} K_\wp(w) = -\kappa^2; \\ \frac{1}{2}, & \text{if } \sup_{w \in \mathcal{M}} K_\wp(w) = 0; \\ \frac{\sin[\kappa\rho_1]}{2\kappa\rho_1}, & \text{if } \sup_{w \in \mathcal{M}} K_\wp(w) = \kappa^2. \end{cases}$$

It is well-known the following fact, a simply connected minimal surface $\Sigma \subset \mathbf{R}^3$, lying over a simply connected domain $\Omega \neq \mathbf{R}^2$, can be parameterized by a conformal harmonic parametrization (Weierstrass-Enneper parametrization) $w = f(z) = (f_1(z), f_2(z), f_3(z)) : D(0, 1) \rightarrow \Sigma$, where $D(0, 1)$ is the unit disk. Moreover Σ is a Riemann surface with a conformal metric $\wp = (|g'(z)| + |h'(z)|)|dz|$ with negative Gaussian curvature. Here g and h are certain holomorphic functions in the unit disk $D(0, 1)$ such that $f(z) = g(z) + \overline{h(z)}$ is a harmonic diffeomorphism of the unit disk onto the domain Ω (cf. [3, Chapter 10]).

The conformal modulus of an annulus \mathcal{A} in complex plane or in a Riemann surface is $\text{Mod}(\mathcal{A}) := \log \frac{r_2}{r_1}$, where r_1 and r_2 are inner and outer radii of a circular annulus $A = A(r_1, r_2)$ conformally equivalent to \mathcal{A} .

By using Theorem 1.1 (i.e. its reformulated version Theorem 4.1) and the previous facts we have

Corollary 1.2. *Assume that (Σ, φ) is a simply connected minimal surface other than a whole plane. Then for every geodesic annulus $A_\varphi(\tilde{0}, \rho_1, \rho_2) \subset \Sigma$ there hold the inequality*

$$(1.6) \quad \frac{\rho_2}{\rho_1} > \frac{1}{2} \text{Mod}(A_\varphi(\tilde{0}, \rho_1, \rho_2))^2 + 1,$$

or what is the same for a circular annulus $A = A(\rho_1, \rho_2) \subset \mathbf{R}^2$ and for the exponential map $\exp = \exp_{\tilde{0}}$ (cf. subsection 2.2), we have

$$\exp(\text{Mod}(A)) > \frac{1}{2} \text{Mod}(\exp(A))^2 + 1.$$

Proof (of Corollary 1.2). Let $w = f(z) = (f_1(z), f_2(z), f_3(z)) : D(0, 1) \rightarrow \Sigma$ be a Weierstrass-Enneper parametrization of a minimal surface $\Sigma \subset \mathbf{R}^3$ such that $f(0) = \tilde{0}$. Let $\hat{A} \subset D(0, 1)$ be the pull-back of $A_\varphi(\tilde{0}, \rho_1, \rho_2)$ under f . Since f is conformal, we obtain that

$$\text{Mod}(\hat{A}) = \text{Mod}(A_\varphi(\tilde{0}, \rho_1, \rho_2)).$$

Since the Gaussian curvature is negative then the function Ψ defined in (1.5) satisfies the inequality $\Psi_\varphi(K) \geq \Psi_\varphi(0) = 1/2$. From Theorem 4.1 we obtain (1.6). In order to obtain the second inequality we only need to point out that $\exp(A(\rho_1, \rho_2)) = A_\varphi(\tilde{0}, \rho_1, \rho_2)$ which follows from the elementary properties of exponential map. \square

Remark 1.3. In connection with Theorem 1.1 (Theorem 4.1), we should notice that the ratio ρ_2/ρ_1 determines the modulus of annulus $A_\varphi(p, \rho_1, \rho_2)$ only in the surface with a flat metric (a metric with zero Gaussian curvature). This means in particular that we cannot obtain in general an inequality involving only the conformal moduli of an annulus and of its harmonic image.

Theorem 1.1 is an extension of the main results in [14, 15, 5, 24] where it is proved the same result but only for the Euclidean, Riemann and Hyperbolic metric on the unit disk. It can be considered as a counterpart of some recent results of Iwaniec, Koh, Kovalev and Onninen [7, 6] where the authors established the existence of harmonic diffeomorphisms between the annuli in complex plane, and of the author [17], where the author established the existence of harmonic diffeomorphisms between annuli on Riemann surface, provided that the conformal modulus of domain is less or equal to the conformal modulus of the target. The proof of Theorem 1.1 is given in section 4 and it depends on the geometry of Riemann surface, more precisely we use a version of Hessian comparison theorem of Yau and Schoen, which is one of most fundamental tools of Riemannian geometry. We make use as well of a comparison principle due to Osserman. Two of the key steps of the proof are: computation of the Laplacian of the distance function of a harmonic map in geodesic polar coordinates with variable Gaussian curvature and comparison of that Laplacian with the Laplacian of distance function of a harmonic mapping in geodesic polar coordinates with constant Gaussian curvature.

2. PRELIMINARIES

2.1. Harmonic mappings between Riemann surfaces. Let (\mathcal{N}, \bar{h}) and (\mathcal{M}, \wp) be Riemann surfaces with conformal metrics \bar{h} and \wp , respectively. If a mapping $f : (\mathcal{N}, \bar{h}) \rightarrow (\mathcal{M}, \wp)$ is C^2 , then f is said to be harmonic (\wp -harmonic) if

$$(2.1) \quad f_{z\bar{z}} + (\log \wp^2)_w \circ f f_z f_{\bar{z}} = 0,$$

where z and w are the local parameters on \mathcal{N} and \mathcal{M} respectively (see [11]). Also f satisfies (2.1) if and only if its Hopf differential

$$(2.2) \quad \Psi[f] = \wp^2 \circ f f_z \overline{f_z}$$

is a holomorphic quadratic differential on \mathcal{N} .

Let D be a domain in \mathbb{C} and \wp be a conformal metric in D . The Gaussian curvature of the smooth metric \wp is given by

$$K_\wp = -\frac{\Delta \log \wp}{\wp^2}.$$

2.2. Geodesic polar coordinates. Assume that (\mathcal{M}, \wp) is a Riemannian surface with conformal metric $\wp(w)|dw|$ whose Gaussian curvature K is bounded from above by a constant $\sup K = \pm \kappa^2$. The inner product in the tangent space $T_w \mathcal{M}$ is given by $g_w(\zeta, \xi) := \wp^2(w) \langle \zeta, \xi \rangle$. The distance function d_\wp is defined as follows

$$d_\wp(p, q) := \inf_{p, q \in \gamma} \int_\gamma \wp(w)|dw|,$$

where the infimum runs over all rectifiable Jordan arcs $\gamma \subset \mathcal{M}$ connecting p and q . The disk on the Riemannian surface with the center $p \in \mathcal{M}$ and the radius $\rho_p > 0$ is defined as

$$B_\wp(p, \rho_p) = \{q \in \mathcal{M} : d_\wp(p, q) < \rho_p\}.$$

The disc $B_\wp(p, \rho_p)$ is called geodesic, if for any $a, b \in B_\wp(p, \rho_p)$ it exists a geodesic curve $c \subset B_\wp(p, \rho_p)$ joining a and b . Equivalently, $B_\wp(p, \rho_p)$ is the diffeomorphic image of the Euclidean open disk $D(0, \rho_p)$ under the exponential map

$$\exp : D(0, \rho_p) \rightarrow B_\wp(p, \rho_p) \subset \mathcal{M}.$$

Let

$$(2.3) \quad ds^2 = d\rho^2 + G^2(\rho, \theta)d\theta^2$$

be the metric in geodesic polar coordinates on the geodesic disk $B_\wp(p, \rho_p)$, $p \in \mathcal{M}$, where $\rho_p > 0$ and for $\sup K = \kappa^2$ it satisfies the additional condition

$$(2.4) \quad \rho_p \leq \frac{\pi}{2\kappa}.$$

The condition (2.4) is related to the comparison theorem of Morse-Schoenberg, which implies the local diffeomorphic behavior of exponential map, and the last fact is used by Jost in the proof of [11, Theorem 2.1] for the existence of a geodesic ball. The geodesic annulus with the inner and outer radii ρ_1 and ρ_2 is defined as

$$A_\wp(p, \rho_1, \rho_2) = \{q \in \mathcal{M} : \rho_1 < d_\wp(p, q) < \rho_2\},$$

where ρ_2 satisfies (2.4). We shall use the following well-known facts, which follows from (2.3)

$$(2.5) \quad g_w \left(\frac{\partial}{\partial \rho} \Big|_w, \frac{\partial}{\partial \rho} \Big|_w \right) = 1$$

$$(2.6) \quad g_w \left(\frac{\partial}{\partial \theta} \Big|_w, \frac{\partial}{\partial \theta} \Big|_w \right) = G^2(\rho, \theta)$$

$$(2.7) \quad g_w \left(\frac{\partial}{\partial \theta} \Big|_w, \frac{\partial}{\partial \rho} \Big|_w \right) = 0.$$

For the definition of the above concepts and detailed study of the Riemannian metrics we refer to [9, Chapter 1].

2.3. Rotationally symmetric metrics and harmonic mappings. The following lemma is essentially proved in the author's paper [14], but for the completeness and for the future reference we include its simplified proof here.

Lemma 2.1. *Assume that \wp is a rotationally symmetric metric, i.e. $\wp(w) = h(|w|)$ for some smooth real function h defined in a segment $[0, a]$. Let f be a \wp -harmonic mapping of the unit disk onto $D_\wp(p, s)$ and assume that $(\rho(z), \theta(z))$ are geodesic polar coordinates of the point $w = f(z)$. Then*

$$G(\rho, \theta) = \eta(\rho)/\eta'(\rho)$$

where $\eta^{-1}(|z|) = d_\wp(p, z)$ and

$$(2.8) \quad \Delta \rho(z) = \frac{1}{2} \frac{\partial G^2}{\partial \rho} |\nabla \theta|^2.$$

Proof. Let g be the inverse of the function $s \mapsto d_\wp(s, 0)$. Then we have

$$\rho = \int_0^{g(\rho)} h(t) dt.$$

Thus

$$(2.9) \quad 1 = g'(\rho) \cdot h(g(\rho)),$$

and

$$(2.10) \quad \frac{h'}{h} = -\frac{g''}{g'^2}.$$

Therefore the metric of the surface can be expressed as

$$ds^2 = \wp(w)^2 |dw|^2 = h(\rho)^2 |d(g(\rho)e^{i\theta})|^2 = d\rho^2 + G^2(\rho) d\theta^2,$$

where

$$(2.11) \quad G^2(\rho) = h^2(g(\rho))g^2(\rho).$$

If $w = f(z)$, is a twice differentiable, then

$$f(z) = g(\rho(z))e^{i\theta}.$$

Now we have

$$\begin{aligned} w_x &= (g' \rho_x + i g \theta_x) e^{i\theta}, \\ w_y &= (g' \rho_y + i g \theta_y) e^{i\theta}, \end{aligned}$$

and thus

$$(2.12) \quad w_{xx} = (g'' \rho_x^2 + g' \rho_{xx} + 2i g' \rho_x \theta_x + i g \theta_{xx} - g \theta_x^2) e^{i\theta},$$

$$(2.13) \quad w_{yy} = (g'' \rho_y^2 + g' \rho_{yy} + 2i g' \rho_y \theta_y + i g \theta_{yy} - g \theta_y^2) e^{i\theta},$$

and

$$(2.14) \quad w_z w_{\bar{z}} = \frac{1}{4} (w_x^2 + w_y^2).$$

Assume now that w is harmonic. By applying (2.12), (2.13), (2.14) and (2.1) in view of

$$(2.15) \quad \partial_w \log \wp^2(w) = \frac{h'(|w|) \overline{w}}{h(|w|) |w|},$$

it follows that

$$\begin{aligned} & (g'' |\nabla \rho|^2 + g' \Delta \rho + 2i g' \langle \nabla \rho, \nabla \theta \rangle + i g \Delta \theta - g |\nabla \theta|^2) e^{i\theta} \\ & + \frac{h'(g(\rho)) e^{-i\theta}}{h(g(\rho))} \left(g'^2 |\nabla \rho|^2 + 2i g' \langle \nabla \rho, \nabla \theta \rangle - g^2 |\nabla \theta|^2 \right) e^{2i\theta} = 0. \end{aligned}$$

Thus

$$\begin{aligned} & (g'' |\nabla \rho|^2 + g' \Delta \rho + 2i g' \langle \nabla \rho, \nabla \theta \rangle + i g \Delta \theta - g |\nabla \theta|^2) \\ & + \frac{h'(g(\rho))}{h(g(\rho))} \left(g'^2 |\nabla \rho|^2 + 2i g' \langle \nabla \rho, \nabla \theta \rangle - g^2 |\nabla \theta|^2 \right) = 0. \end{aligned}$$

Therefore

$$(2.16) \quad 2g' \langle \nabla \rho, \nabla \theta \rangle + g \Delta \theta + 2 \frac{h'(g(\rho)) g'(\rho)}{h(g(\rho))} \langle \nabla \rho, \nabla \theta \rangle = 0$$

and

$$(2.17) \quad (g'' |\nabla \rho|^2 + g' \Delta \rho - g |\nabla \theta|^2) + \frac{h'(g(\rho))}{h(g(\rho))} \left(g'^2 |\nabla \rho|^2 - g^2 |\nabla \theta|^2 \right) = 0.$$

Combining (2.10) and (2.16) it follows that

$$(2.18) \quad g' \Delta \rho = \left(g(\rho) + \frac{h'(g(\rho))}{h(g(\rho))} g^2 \right) |\nabla \theta|^2.$$

From (2.9) we obtain

$$(2.19) \quad \Delta \rho = g(\rho) [h(g(\rho)) + h'(g(\rho)) g(\rho)] |\nabla \theta|^2$$

i.e.

$$\Delta \rho = \frac{1}{2} \frac{\partial G^2}{\partial \rho} |\nabla \theta|^2.$$

□

The proof of the previous lemma contains in particular the following lemma.

Lemma 2.2. *For $\kappa \neq 0$, the Gaussian curvature of the metric*

$$\hat{\phi} = \frac{2|dz|}{\kappa(1 \pm |z|^2)} \text{ is } K(z) = \pm \kappa^2 \neq 0,$$

while for $\kappa = 0$, the corresponding metric is the Euclidean one: $\hat{\phi}(w) = |dw|$. Its distance function is

$$r(|z|) = \begin{cases} \frac{2}{\kappa} \tan^{-1}(|z|), & \text{if } K(z) = \kappa^2; \\ \frac{2}{\kappa} \tanh^{-1}(|z|), & \text{if } K(z) = -\kappa^2; \\ |z|, & \text{if } K(z) = 0. \end{cases}$$

Therefore

$$\hat{g}(\rho) = \begin{cases} \tan \frac{2\rho}{\kappa}, & \text{if } K(z) = \kappa^2; \\ \tanh \frac{2\rho}{\kappa}, & \text{if } K(z) = -\kappa^2; \\ \rho, & \text{if } K(z) = 0. \end{cases}$$

Thus for a constant curvature surface we have

$$(2.20) \quad \hat{G}(\rho, \theta) = \begin{cases} \frac{\sin(\kappa\rho)}{\kappa}, & \text{if } K(z) = \kappa^2; \\ \rho, & \text{if } K(z) = 0; \\ \frac{\sinh(\kappa\rho)}{\kappa}, & \text{if } K(z) = -\kappa^2. \end{cases}.$$

The following function is well-known in the Riemann geometry, especially in connection with Hessian and Laplacian comparison theorem.

$$h_c(r) = \begin{cases} \sqrt{c} \cot(\sqrt{c}r), & \text{if } c > 0; \\ \frac{1}{r}, & \text{if } c = 0; \\ \sqrt{-c} \coth(\sqrt{-c}r), & \text{if } c < 0. \end{cases}$$

The main result lies on the inequality (3.1), which will be proved by using the following Hessian comparison theorem due to Yau and Schoen [23].

Proposition 2.3 (Hessian Comparison Theorem). *(see [23] cf. [2, Theorem 3.2]). Let \mathcal{M} be a Riemannian manifold and $p, q \in \mathcal{M}$ be such that there is a minimizing unit speed geodesic γ joining p and q , and let $r(x) = \text{dist}(p, x)$ be the distance function to p . Let $K_\gamma \leq c$ be the radial sectional curvatures of \mathcal{M} along γ . If $c > 0$ assume $r(q) < \frac{\pi}{2\sqrt{c}}$. Then we have*

$$(2.21) \quad \text{Hess } r(x)(\gamma', \gamma') = 0$$

and

$$(2.22) \quad \text{Hess } r(x)(X, X) \geq h_c(r(x))|X|_g^2,$$

where $X \in T_x \mathcal{M}$ is perpendicular to $\gamma'(r(x))$. Here $\text{Hess}(r(x))$ is the Hessian matrix of $r(x)$.

Now we formulate a comparison theorem of Osserman [20].

Theorem 2.4 (Comparison Theorem). *Let ds^2 and $d\hat{s}^2$ be metrics given in geodesic polar coordinates by*

$$\begin{aligned} ds^2 &= d\rho^2 + G^2(\rho, \theta)^2 d\theta^2 \\ d\hat{s}^2 &= d\rho^2 + \hat{G}^2(\rho, \theta)^2 d\theta^2. \end{aligned}$$

If the Gaussian curvatures satisfies

$$(2.23) \quad K(\rho, \theta) \leq \hat{K}(\rho, \theta), \quad 0 < \rho < \rho_0,$$

then

$$(2.24) \quad \frac{1}{G^2} \frac{\partial G^2}{\partial \rho} \geq \frac{1}{\hat{G}^2} \frac{\partial \hat{G}^2}{\partial \rho}$$

and

$$(2.25) \quad G^2(\rho, \theta) \geq \hat{G}^2(\rho, \theta), \quad 0 < \rho < \rho_0.$$

3. THE KEY LEMMAS

Lemma 3.1. *Let $w = f(z)$ be a harmonic mapping of an open subset $\Omega \subset \mathbf{R}^2$ into the geodesic disk $B_g(\tilde{0}, p) \subset (\mathcal{M}, \varphi)$ of the Riemannian surface (\mathcal{M}, g) with a Gaussian curvature $K \leq c$, $\tilde{0} \in \mathcal{M}$. Assume that $ds^2 = dr^2 + G^2(r, \theta)d\theta^2$ is the metric in geodesic polar coordinates in tangential space $T_{\tilde{0}}\mathcal{M}$ of the surface \mathcal{M} . Let r be the distance function from the fixed point $\tilde{0} \in \mathcal{M}$. Define $\rho(z) = r(f(z))$. Then we have the following inequality*

$$(3.1) \quad \Delta \rho \geq h_c(\rho) G^2(\rho, \theta) |\nabla \theta|^2,$$

where Δ and ∇ are standard Euclidean Laplacian and Euclidean gradient respectively.

Proof. In order to obtain Laplacian of $\rho(z)$ we use an approach similar to that on the book of Schoen and Yau ([22, P. 176]) (cf. [11, Eq. 5.1.2]).

Viewing ρ as a composite function and applying the chain rule, by using the notation $z = (z^1, z^2) = x + iy$ and assuming that $w = (w^1, w^2)$ are Riemann normal coordinates on \mathcal{M} , we get

$$\begin{aligned} \Delta d(w(z), \tilde{0}) &= \sum_{i, \alpha} \left(\rho_i \frac{\partial w^i}{\partial z^\alpha} \right)_\alpha \\ &= \sum_{i, j, \alpha} \rho_{ij} \frac{\partial w^i}{\partial z^\alpha} \frac{\partial w^j}{\partial z^\alpha} + \sum_i \rho_i \Delta w^i. \end{aligned}$$

Here the lower indices denote the partial derivatives. Since w is harmonic, and w^1, w^2 are normal coordinates, then the second term vanishes. Thus we obtain the formula

$$(3.2) \quad \Delta d(w(z), \tilde{0}) = \sum_{\alpha \in \{x, y\}} \text{Hess}(\rho)(\nabla_\alpha w, \nabla_\alpha w).$$

Moreover in geodesic polar coordinates we have $w = \rho e^{i\theta}$ and therefore

$$\nabla_x w = \nabla_x(\rho e^{i\theta}) = \rho_x e^{i\theta} - i\rho \theta_x e^{i\theta}$$

and

$$\nabla_y w = \nabla_y(\rho e^{i\theta}) = \rho_y e^{i\theta} - i\rho \theta_y e^{i\theta}.$$

The last two relations imply

$$(3.3) \quad \Delta d(w(z), \tilde{0}) = |\nabla \rho|^2 \text{Hess}(\rho)(e^{i\theta}, e^{i\theta}) + |\nabla \theta|^2 \text{Hess}(\rho)(\rho i e^{i\theta}, \rho i e^{i\theta}).$$

Since Gaussian curvature $K \leq \pm \kappa^2$, then the radial sectional curvature along $\gamma(t) = \exp_{\tilde{0}}(te^{i\theta})$ is also bounded by $\pm \kappa^2$. For $w = f(z)$ we have

$$(3.4) \quad \left. \frac{\partial}{\partial \theta} \right|_w = \rho i e^{i\theta}$$

and

$$(3.5) \quad \left| \left. \frac{\partial}{\partial \theta} \right|_w \right|^2 = \left\langle \left. \frac{\partial}{\partial \theta} \right|_w, \left. \frac{\partial}{\partial \theta} \right|_w \right\rangle = g\left(\left. \frac{\partial}{\partial \theta} \right|_w, \left. \frac{\partial}{\partial \theta} \right|_w\right) = G^2(\rho, \theta).$$

Because of (2.21) and

$$(3.6) \quad \gamma'(\rho(z)) = \nabla \rho = e^{i\theta},$$

we obtain

$$(3.7) \quad |\nabla \rho|^2 \text{Hess}(\rho)(e^{i\theta}, e^{i\theta}) = 0.$$

By using now (2.22) and (3.3) – (3.7) we obtain

$$\begin{aligned} \Delta \rho &= \Delta d(w(z), \tilde{0}) \\ &= |\nabla \theta|^2 \text{Hess}(\rho)(\rho i e^{i\theta}, \rho i e^{i\theta}) \\ &= |\nabla \theta|^2 \text{Hess}(\rho) \left(\left. \frac{\partial}{\partial \theta} \right|_w, \left. \frac{\partial}{\partial \theta} \right|_w \right) \\ &\geq h_\kappa(\rho) |\nabla \theta|^2 \left| \left. \frac{\partial}{\partial \theta} \right|_w \right|^2 \\ &= h_\kappa(\rho) G^2(\rho, \theta) |\nabla \theta|^2, \end{aligned}$$

which yields (3.1). \square

By using comparison theorem of Osserman we obtain

Lemma 3.2. *Under the condition of Lemma 3.1, for the Riemannian surfaces (\mathcal{M}, \wp) with Gaussian curvature K bounded from above we have the following sharp inequality*

$$(3.8) \quad \Delta \rho \geq \psi_\wp(\rho) |\nabla \theta|^2,$$

where

$$(3.9) \quad \psi_\wp(\rho) = \begin{cases} \frac{\sinh[\kappa \rho]}{\kappa}, & \text{if } \sup K = -\kappa^2; \\ \rho, & \text{if } \sup K = 0; \\ \frac{\sin[\kappa \rho]}{\kappa}, & \text{if } \sup K = \kappa^2. \end{cases}$$

Proof. Let $ds^2 = d\rho^2 + G^2(\rho, \theta) d\theta^2$ and $d\hat{s}^2 = d\rho^2 + \hat{G}^2(\rho, \theta) d\theta^2$ be metrics given in geodesic polar coordinates that present the metric \wp and the constant curvature metric $\hat{\wp}$ ($K \equiv \pm \kappa^2$). Combining (3.1), (2.25) and (2.20) we arrive at (3.8). From Lemma 2.1, we conclude that the inequality (3.8) reduces to an equality for constant curvature metrics. This proves the sharpness of (3.8). \square

4. THE PROOF OF MAIN RESULT

We are now prepare to prove Theorem 1.1, i.e. its slight reformulation:

Theorem 4.1. *Assume that \mathcal{N} and \mathcal{M} are Riemann surfaces and let \mathcal{A} be a doubly connected domain in \mathcal{N} . If there exists a harmonic homeomorphism f between annuli $\mathcal{A} \subset \mathcal{N}$ and $A_\varphi(p, \rho_1, \rho_2) \subset \mathcal{M}$, then we have*

$$(4.1) \quad \frac{\rho_2}{\rho_1} \geq \Psi_\varphi \text{Mod}(\mathcal{A})^2 + 1,$$

where

$$\Psi_\varphi = \begin{cases} \frac{\sinh[\kappa\rho_1]}{2\kappa\rho_1}, & \text{if } \sup K = -\kappa^2; \\ \frac{1}{2}, & \text{if } \sup K = 0; \\ \frac{\sin[\kappa\rho_1]}{2\kappa\rho_1}, & \text{if } \sup K = \kappa^2. \end{cases}$$

Recall that the case $\sup K = \kappa^2$, is subject of the a priory condition (2.4) for ρ_2 i.e. of $\rho_2 \leq \pi/(2\kappa)$.

In order to prove Theorem 4.1 we make use of the following proposition.

Proposition 4.2. *Let $w = \rho e^{i\theta}$ be a C^1 surjection between the rings $A(r_1, r_2)$ and $A(s_1, s_2)$ of the complex plane $\mathbf{C} = \mathbf{R}^2$. Then*

$$(4.2) \quad \int_{r_1 \leq |z| \leq r_2} |\nabla \theta|^2 dx dy \geq 2\pi \log \frac{r_2}{r_1}.$$

For its proof see for example [15].

Proof of Theorem 4.1. Assume for a moment that f is smooth up to the boundary and \mathcal{A} is the circular annulus $A(r_1, r_2)$ and f maps the inner boundary onto the inner boundary. By applying Green's formula for ρ on $\{z : r_1 \leq |z| \leq \sigma\}$, $r_1 < \sigma < r_2$, we obtain

$$\int_{|z|=\sigma} \frac{\partial \rho(z)}{\partial \sigma} |dz| - \int_{|z|=r_1} \frac{\partial \rho(z)}{\partial \sigma} |dz| = \int_{r_1 \leq |z| \leq \sigma} \Delta \rho d\mu.$$

Here $d\mu$ is the twodimensional Lebesgue measure. Since $\frac{\partial \rho}{\partial \sigma} \geq 0$ for $|z| = r_1$ we obtain

$$\int_{|z|=\sigma} \frac{\partial \rho(z)}{\partial \sigma} |dz| \geq \int_{r_1 \leq |z| \leq \sigma} \Delta \rho d\mu.$$

By applying (3.8) and (4.2), and using the fact that the function ψ_φ defined in (3.9) is an increasing function in ρ , we obtain

$$\begin{aligned} \int_{|z|=\sigma} \frac{\partial \rho(z)}{\partial \sigma} |dz| &\geq \int_{r_1 \leq |z| \leq \sigma} \Delta \rho d\mu \\ &\geq \int_{r_1 \leq |z| \leq \sigma} \psi_\varphi(\rho) |\nabla \theta|^2 d\mu \\ &\geq \psi_\varphi(\rho_1) \int_{r_1 \leq |z| \leq \sigma} |\nabla \theta|^2 d\mu \\ &\geq 2\pi \psi_\varphi(\rho_1) \log \frac{\sigma}{r_1}. \end{aligned}$$

It follows that

$$\sigma \frac{\partial}{\partial \sigma} \int_{|\zeta|=1} \rho(\sigma \zeta) |d\zeta| \geq 2\pi \psi_{\wp}(\rho_1) \log \frac{\sigma}{r_1}.$$

Dividing by σ and integrating over $[r_1, r_2]$ by σ the previous inequality, we get

$$\begin{aligned} \int_{|\zeta|=1} \rho(r_2 \zeta) |d\zeta| - \int_{|\zeta|=1} \rho(r_1 \zeta) |d\zeta| \\ \geq \pi \psi_{\wp}(\rho_1) \log^2 \frac{r_2}{r_1} \end{aligned}$$

i.e.

$$(4.3) \quad 2\pi(\rho_2 - \rho_1) \geq \pi \psi_{\wp}(\rho_1) \log^2 \frac{r_2}{r_1}.$$

If f is not smooth up to the boundary, then instead of f we consider the mapping f_{ϵ} , $0 < \epsilon < (\rho_2 - \rho_1)/2$ constructed as follows. Let $A_{\wp}(p, \rho_1 + \epsilon, \rho_2 - \epsilon) \subset A_{\wp}(p, \rho_1, \rho_2)$ and define $A_{\epsilon} = f^{-1}(A_{\wp}(p, \rho_1 + \epsilon, \rho_2 - \epsilon))$. Take a conformal mapping $\phi_{\epsilon} : A(r_{\epsilon}, R_{\epsilon}) \rightarrow A_{\epsilon}$ and define $f_{\epsilon} = f \circ \phi_{\epsilon}$. Then $f_{\epsilon} : A_{\epsilon} \rightarrow A_{\wp}(p, \rho_1 + \epsilon, \rho_2 - \epsilon)$ is a \wp -harmonic diffeomorphism because its Hopf differential

$$\Psi[f_{\epsilon}] = \Psi[f](\phi_{\epsilon})(\psi'_{\epsilon}(z))^2,$$

is holomorphic. Moreover

$$\lim_{\epsilon \rightarrow 0} \log \frac{R_{\epsilon}}{r_{\epsilon}} = \text{Mod}(\mathcal{A}).$$

Then we apply the previous case and let $\epsilon \rightarrow 0$ in order to obtain (4.1) if f maps the inner boundary onto the inner boundary. If f maps the inner boundary onto the outer boundary, then we consider the composition of f by the conformal mapping $\phi(z) = r_1 r_2 / z$. \square

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FACULTY OF NATURAL SCIENCES AND MATHEMATICS, UNIVERSITY OF MONTENEGRO,
 CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO
E-mail address: davidk@ac.me